The Consecutive Multiprocessor Job Scheduling Problem

Yossi Bukchin[1]

Department of Industrial Engineering,

Tel-Aviv University, Ramat Aviv 6997801, Tel Aviv, Israel

Email: bukchin@tau.ac.il

Tal Raviv

Department of Industrial Engineering,

Tel-Aviv University, Ramat Aviv 6997801, Tel Aviv, Israel

Email: talraviv@eng.tau.ac.il

Ilya Zaides

Department of Industrial Engineering,

Tel-Aviv University, Ramat Aviv 6997801, Tel Aviv, Israel

Email: ilyazaides@mail.tau.ac.il

Submitted: September 2018

Revised: May 2019

[1] Corresponding author

The Consecutive Multiprocessor Job Scheduling Problem

Abstract

We study a variant of the multiprocessor job scheduling problem, where jobs are processed by several identical machines. The machines are ordered in a sequence, and each job is processed by several consecutive machines simultaneously. The jobs are characterized by their processing time, the number of required consecutive machines, and their ready time. The objective function is to minimize the sum of general functions defined over the completion time of each job. This study is motivated by a real problem in the semiconductor industry. We present a time-indexed integer programming and a constraint programming formulations for the problem and demonstrate their applicability through an extensive numerical study and an industrial case study.

Keywords: multiprocessor job scheduling, integer programming, constraint programming

1. Introduction and preliminaries

We study a scheduling problem that can be stated as follows: A system with an ordered set of m identical machines, located one next to the other, is used to process a set of n jobs. The processing of each job occupies a set of several consecutive machines simultaneously, while each machine can process at most one job at a time. Each job is characterized by its processing time, number of required consecutive machines, ready time, and, possibly, due date. For example, a job that requires three machines, with a time duration of five hours, may be processed *simultaneously* on machines 2, 3 and 4, starting at $t = 7$ and ending at $t = 12$. The objective function is to minimize the sum of general functions defined over the completion time of each job. This includes minimizing the number of tardy jobs, and minimizing the total tardiness, as special cases. The problem is named the Consecutive Multiprocessor Job Scheduling Problem. It is a generalization of some known NP-hard and inapproximable scheduling problems, including minimizing the number of tardy jobs on parallel machines, see for example (Pinedo, 2012).

This study is motivated by a real-world application of allocating computing resources during the development of new integrated circuits in the semiconductors industry. The computing resources are hardware emulators that are being used to save time and costs associated with the verification of new designs before beginning the manufacturing process. Each verification job needs to be processed by several consecutive emulators in the array for a given time. The developers in the various departments specify in advance the time by which each design will be ready for verification. The due dates are dictated by the production plans of the FABs and other managerial considerations. The emulators are expensive and scarce. Thus, it is important to utilize them efficiently. Similar problems arise in different applications, such as allocating berth space and cranes in a seaport.

In this study, we present two formulations for the consecutive multiprocessor job scheduling problem: a time-indexed integer programming (IP) model and a constraint programming (CP) model. In an extensive numerical study, we compare these formulations with state-of-the-art formulations from the literature and demonstrate their effectiveness. Our time-indexed formulation can be viewed as a generalization of previous similar formulations for classic parallel machine scheduling problems, see for example (Sevaux & Thomin, 2001). In addition, we apply our method in a rolling horizon scheme for an Industrial case study and compare its performance with the current practice.

The rest of this paper is organized as follows. In Section 2, we review the current literature and identify the gaps that need to be addressed. In Section 3, we the formulations for the consecutive multiprocessor job scheduling problem. The performance of these formulations are tested with commercial solvers in Section 4, and are compared with a previous formulation that is capable of solving some special cases of the problem. In Section 5, we present the results of an industrial case study where our algorithm is applied in a dynamic setting. Concluding remarks are given in Section 6.

2. Literature review

The problem of scheduling jobs on multiple processors shares similar properties to well-studied problems such as scheduling jobs on parallel machines, the multiprocessor scheduling problem, strip packing and cutting problems, the berth allocation problem (BAP); however, the studied problem differs from the ones discussed in the literature either by the scheduling environment or by the objective function.

(Leonardi & Raz, 2007) studied the parallel processors for minimizing total flow time problem (known as $P/r_i/\sum c_i$). They proved that the problem is not approximable within $O\left(n^{\frac{1}{3}-\epsilon}\right)$ for any $\epsilon > 0$. This problem is a special case of our problem since we consider jobs that require more than one machine at a time. We conclude that the Consecutive Multiprocessor Job Scheduling Problem is strongly NP-hard and is not approximable within a constant. Hence, the motivation for the solution methods presented in this paper.

The general multiprocessor job scheduling has been studied in (Bianco, Blazewicz, Dell'Olmo, & Drozdowski, 1995) and (Chen & Lee, 1999). In the generalized problem, each job can be processed by multiple alternative sets of machines, not necessarily with identical cardinality. For example, job i can be processed either by machines $\{2, 4\}$ or by machines $\{3, 4, 5\}$. Hence, all alternatives have to be explicitly presented. In (Chen & Lee, 1999), a two phase heuristic is proposed for solving the problem of a general multiprocessor with the objective of minimizing the makespan, as follows: an assignment phase, in which alternatives are being chosen for each job, and a scheduling phase, in which the job ordering is being determined. (Baptiste, 2003) addresses a special case of the general multiprocessors scheduling problem, when all jobs have the same processing time and each job has its own ready time and size (required number of machines). He shows that both completion time and makespan minimization problems can be solved in polynomial time using a dynamic programming algorithm. (Huang, Chen, & Wang, 2007) present a linear time 1.5 approximation algorithm for minimizing the makespan on four machines.

Note that in our model, each job has its own size, which is the required number of *consecutive* machines, and hence, the problem can be expressed in a more concise manner. Cutting rectangular elements and strip packing problems, see for example, (Martello, Monaci, & Vigo, 2003) have similar structure to our problem but differ in their objective function, constraints and in the fact that the size of each job is not necessarily discrete.

Some variations of the berth allocation problem (BAP) are highly related to the general multiprocessor scheduling problems. This problem deals with allocating berth space and cranes in seaports to arriving vessels that have to be loaded and unloaded. As described in (Bierwirth & Meisel, 2010), there are three variations of this problem, as follows:

1. Discrete case (BAP-D). The quay is divided into separate sections, where each section (berth) can serve one vessel at a time (Imai, Chia, Etsuko, & Papadimitriou, 2008), (Imai, Nishimura, & Papadimitriou, 2001).
2. Continuous case (BAP-C). In this case, a vessel can berth anywhere along the quay. Consequently, the quay can serve a number of vessels simultaneously (Park & Kim, 2005) (Imai, Sun, Nishimura, & Papadimitriou, 2005).
3. Hybrid case (BAP-H). The quay is divided into separate sections, as each section is continuous and may serve several vessels, and each vessel may occupy several sections (Umang, Bierlaire, & Vacca, 2013).

The first case (BAP-D) is analogous to the *classic* parallel machine problem, as each vessel\job can be served by a single section (machine). The second case, (BAP-C) is analogous to our problem, if the size of the vessels (jobs) and the length of the quay (sequence of machines) are integers. Our problem is also a special case of the third case (BAP-H), where no two vessels are small enough to fit into one quay section.

The similarity between our problem and BAP-C can also be explained graphically. Each vessel/job is represented by a rectangle, with dimensions of time and space (which is equivalent to multiple machines in the discrete case). These rectangles are then located on a two-dimensional space without overlapping, while considering some objective function. An example of such a solution is given in Figure 1.

(Moon, 2000) suggests a MILP formulation and a heuristic method for the BAP-C, which minimizes a cost function consisting of a vessel berth position relative to the requested and the cost of tardiness. The BAP-C problem with the criterion of minimizing the total flow time of the vessels is addressed in (Guan & Cheung, 2004). A relative position formulation (RPF) and position assignment formulation (PAF) are proposed for solving the problem. Additionally, a special tree search procedure is constructed to create an optimal solution. A comparison between the proposed tree search method and RPF is made, and while the tree search performs better than RPF, it is still highly dependent on the number of vessels (or jobs in our case). Two extensions of this berth allocation model are studied in (Ernst, Oguz, Singh, & Taherkhani, 2017) and (Agra & Oliveira, 2018). The former considers high tide and the latter combines the crane assignment to vessels.

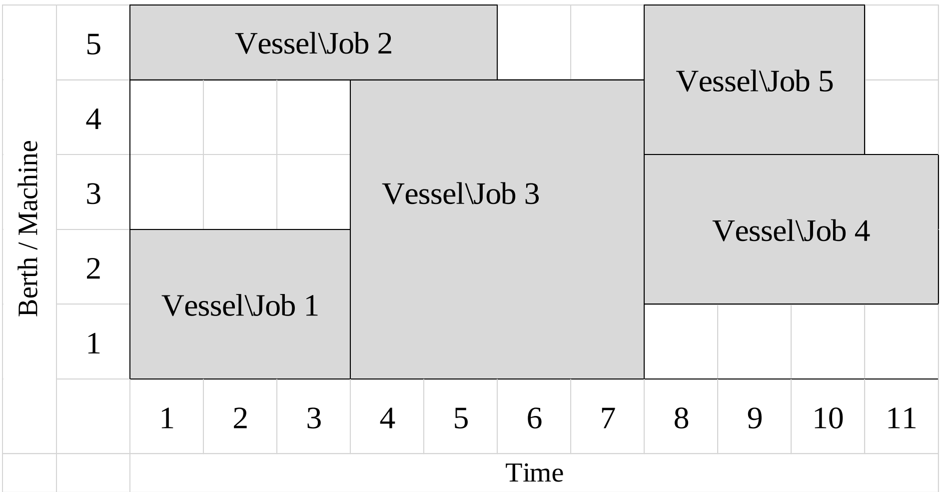


Figure 1. Schedule example

The contribution of our study with respect to the state of the art of the equivalent BAP-C problem is as follows. First, we present new integer programming (IP) and constraint programming (CP) models, which perform well compared to the current existing formulation of (Guan & Cheung, 2004), when solved with a commercial solver (CPLEX in our case). Second, our model is more general in the objective function, as it is capable of optimizing any objective function that is separable in the completion time of the jobs, including nonlinear functions. Such objective functions may be found in various applications in both the contexts of machine and berth scheduling. For example, when the processing costs vary over the hours of the day due to time-of-use electricity tariffs or different labor costs at various hours of the day/week. Finally, we implement our model in a rolling horizon setting to simulate scheduling of test jobs on hardware emulators. Our simulation is based on real-life datasets obtained from a major semiconductor company. We show that our algorithm performs well for this real-world problem, and significantly outperforms the current policy.

3. Mathematical formulations

In this section, we present a time-indexed integer programming formulation and a constraint programming formulation for the consecutive multiprocessor job scheduling problem. In addition, a benchmark mixed integer formulation, inspired by the model of (Guan & Cheung, 2004), was developed and is presented in Appendix A. For each of our modeling techniques, we start with an important special case, which is the minimization of the number of tardy jobs and continue to the general objective function that captures many well-studied scheduling problems with general ready times and due dates. Both objectives were inspired by the real-world case study, mentioned above. Let us first introduce some notation for the parameters of the problem.

n	number of jobs
m	number of machines
p_i	processing time of job $i = 1, \dots, n$
s_i	number of consecutive machines required for the processing of job i (aka size of job i)
r_i	ready time of job i, note that since the time periods are discrete and indexed from 1, $r_i = t$ implies that the earliest period on which the jobs can start is $t + 1$.
d_i	due date of job i
T	the length of the planning horizon.

Throughout the paper, we assume that all the numerical parameters are integers. Based on this assumption, one can assume, without loss of generality, that the starting and completion times of the job are all integers. The planning horizon is divided into T discrete periods where Job i becomes available at the end of period r_i and is due by the end of period d_i.

3.1. Time-indexed integer programming formulation

We first introduce a time-indexed integer programming formulation, which minimizes the number of tardy jobs. To this end, we define the following decision variable:

x_{ijt}	A binary decision variable that equals one if job i is scheduled to start on machines $j, \dots, j + s_1 - 1$ at time t. Both the periods and the machines are indexed from one.

For the sake of compact presentation of the model we define the following parameters:

α_{ij}	The lowest starting machine index possible for job i given it occupies machine j. $\alpha_{ij} = max\{1, j - s_i + 1\}$.
β_{it}	The earliest starting period for job i given it is still being processed at period t. $\beta_{it} = max\{r_i + 1, t - p_i + 1\}$
γ_{it}	The latest starting period for job i given that it is still being processed at period t. $\gamma_{it} = \min\{d_i - p_i + 1, t\}$

Model IP-U

$$\max\left\{\sum_{i=1}^{n}\sum_{j=1}^{m-s_i+1}\sum_{t=r_i+1}^{d_i-p_i+1} x_{ijt}\right\} \tag{1}$$

Subject to:

$$\sum_{j=1}^{m-s_i+1}\sum_{t=r_i+1}^{d_i-p_i+1} x_{ijt} \leq 1 \qquad i = 1,\ldots,n \tag{2}$$

$$\sum_{i=1}^{n}\sum_{j'=\alpha_{ij}}^{j}\sum_{t'=\beta_{it}}^{\gamma_{it}} x_{ij't'} \leq 1 \qquad \begin{array}{l} j = 1,\ldots,m \\ t = 1,\ldots,\max\limits_{i} d_i \end{array} \tag{3}$$

$$x_{ijt} \in \{0,1\} \qquad \begin{array}{l} i = 1,\ldots,n \\ j = 1,\ldots,m-s_i+1, \\ t = r_i+1,\ldots,d_i-p_i+1 \end{array} \tag{4}$$

The objective function (1) maximizes the total number of jobs that were completed before their due date. This objective is equivalent to minimizing the number of tardy jobs. Constraint (2) assures that each job is processed at most once; no scheduling decisions are made for jobs that cannot meet their due date. Constraint (3) is a non-overlapping constraint, which eliminates the possibility of scheduling multiple jobs on the same machine at the same time. Note that only assignments of jobs to feasible machines and starting times that cover machine j at times t are summed at the left-hand side. Indeed, if $t > d_i$ or $t < r_i$ then $\beta_{it} > \gamma_{it}$. In such case the summation is empty. In (4), the binary decision variables are defined for all possible starting periods of jobs within their allowed period.

A toy example, to demonstrate the non-overlapping constraint (3), is given in Figure 2. Assume that only two jobs are considered, job 1, with $p_1 = 5$, and $s_1 = 2$, and job 2, with $p_2 = 3$, and $s_2 = 4$. Let us consider the overlapping avoidance at $t = 6, j = 5$. In this case, job 1 will use machine 5 in time 6 only when $x_{1jt} = 1$ for any combination of $j \in \{4, 5\}$, and $t \in \{2, \ldots, 6\}$. Similarly, job 2 will use the same slot (machine 5 in time 6) only when $x_{2jt} = 1$ for any combination of $j \in \{2, \ldots, 5\}$, and $t \in \{4, \ldots, 6\}$. Constraint (3) then sums over these ranges for the x variables of the two jobs, bounding the summation by 1, i.e., $\sum_{k=4}^{5}\sum_{t'=2}^{6} x_{1kt'} + \sum_{k=2}^{5}\sum_{t'=4}^{6} x_{2kt'} \leq 1$.

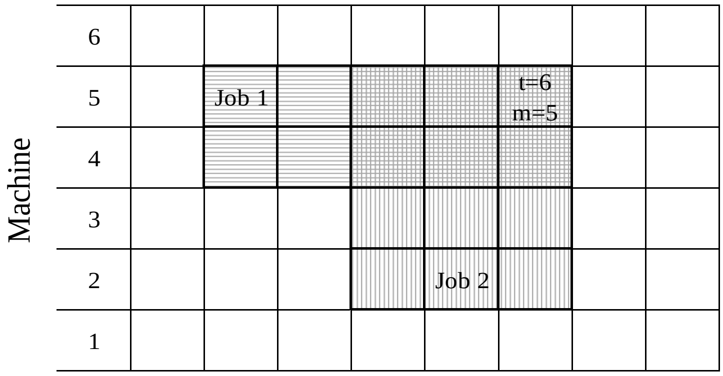


Figure 2. Example for constraint (3)

Next, we extend the model to be useful for a variety of scenarios by assuming a more general objective function. In particular, a parameter c_{it} (typically a cost) is associated with starting each job i at each particular time t (or completing the job at $t + p_i$. It is assumed now that jobs can be started and completed at any time after their ready time and not necessarily before their due dates (if exist), but some cost may be associated with each start time. The extended model is as follows:

Model IP-G

$$\min \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m-s_i-1} \sum_{t=r_i+1}^{T-p_i+1} c_{it} x_{ijt} \right\} \tag{5}$$

Subject to:

$$\sum_{j=1}^{m-s_i+1} \sum_{t=r_i+1}^{T-p_i+1} x_{ijt} = 1 \qquad i = 1, \dots, n \tag{6}$$

$$\sum_{i=1}^{n} \sum_{j'=\alpha_{ij}}^{j} \sum_{t'=\beta_{it}}^{\min\{T-p_i+1,t\}} x_{ij't'} \leq 1 \qquad \begin{array}{l} j = 1, \dots, m \\ t = 1, \dots, T \end{array} \tag{7}$$

$$x_{ijt} \in \{0,1\} \qquad \begin{array}{l} i = 1, \dots, n \\ j = 1, \dots, m - s_i + 1 \\ t = r_i + 1, \dots, T - p_i + 1 \end{array} \tag{8}$$

The objective function (5) of the extended model is to minimize the total cost associated with the start time of all the jobs. Constraint (6) stipulates that each job starts at some time not before its ready time. Constraint (7) eliminates the overlapping of jobs on the same machine similarly to (3), with a different time range in the summation. In (8), the ranges of the decision variables are defined.

Note that model IP-G is general enough to represent various objective functions. Some examples include the sum of the completion times of all the jobs, the sum of functions of the completion times, as well as less studied variants, such as nonlinear tardiness costs and non-monotonous costs that may appear, for example, in the context of time-of-use electricity tariffs. In particular, the problem of minimizing the number of tardy job (as in model IP-U) is obtained by setting $c_{it} = 0$ for all $t = r_i, \dots, d_i - p_i + 1$ and $c_{it} = 1$ for all $t > d_i - p_i + 1$. Similarly, the problem of minimizing the total tardiness is obtained by letting $c_{it} = \max\{0, t + p_i - d_i\}$. However, the IP-U model presented above as a maximization problem is more compact and tractable since the starting time variables are defined only for the periods between the ready time and due date. Specifically, the number of binary variables in the IP-U (resp., IP-G) models is less than $m\sum_{i=1}^{n}(d_i - r_i)$ (resp., mnT). Note that in a typical situation $nT \gg \sum_{i=1}^{n}(d_i - r_i)$. For completeness the number of constraints in IP-U (resp., IP-G) model is $n + m \cdot \max_i d_i$ (resp. $n + mT$).

A variant of the problem that is of interest in our motivating case-study is when the goal is to minimize a weighted sum of the number of tardy jobs and the total tardiness. The penalty associated with each job in this setting may be expressed as follows:

$$w^U \cdot \mathbb{I}_{\{is\ the\ job\ tardy\}} + w^T \cdot \{Tarndiness\}, \tag{9}$$

where $\mathbb{I}_{\{condition\}}$ is an indicator function that returns one if the condition holds and zero otherwise; w^U is the cost for a tardy job and w^T is the penalty for a unit of tardiness. The penalty associated with each job in this setting may be expressed as follows:

$$c_{it} = w^U \cdot \mathbb{I}_{\{t > d_i - p_i\}} + w^T \cdot \max\{0, t - (d_i - p_i + 1)\}. \tag{10}$$

In particular, if our goal is to lexicographically minimize the number of tardy jobs as a primary objective and the total tardiness as a secondary one, we can set $w^U \gg w^T$.

Recall that the length of the planning horizon, T, for model IP-G [(5)-(8)] is not bounded by $\max_{i \in \{1,\dots,n\}} d_i$ as in the case of model IP-U [(1)-(4)] since in this model jobs can be scheduled to be completed after their due date. The value of T dictates the number of decision variables and hence the tractability of the model. If T is externally given, the model does not necessarily admit a feasible solution. In case T is to be determined internally, and the cost associated with the completion time is non-decreasing over time, an upper bound of T in an optimal solution is given by the following:

$$T' = \max_{i\in\{1,\dots,n\}} r_i + \sum_{i=1}^{n} p_i. \tag{11}$$

The upper bound, T', is valid because any schedule that is ended after it must contain periods in which all the machine are idle and all the remaining jobs are ready. If the objective function is non-decreasing in the completion time of all the jobs, it is always possible to shift the jobs that are scheduled after the idle period to an earlier time without increasing the value of the objective function.

However, such a bound may be loose and results in a large number of decision variables in the model. The following procedure can provide a tighter upper bound on T, if our goal is to lexicographically minimize the number of tardy jobs and the total tardiness as a secondary objective. First, the minimal number of tardy jobs, U^*, is obtained by solving the minimization problem of the primary objective (say by solving IP-U). Then, an upper bound on the length of the planning horizon is computed, where $p_{[i]}$ denotes the processing time of the i^{th} longest job, by the following:

$$T'' = \max_{i\in\{1,\dots,n\}} d_i + \sum_{i=1}^{U^*} p_{[i]}. \tag{12}$$

Since neither of the above upper bounds dominates the other, one should use $T = \min(T', T'')$. Note that the upper bound in (12) is valid even if we use an upper bound on U^* rather than its exact value, i.e., if the number of tardy jobs is obtained by some heuristic.

3.2. Constraint programming formulation

Constraint programming (CP) is a solution approach that is based on formulating problems as constraint satisfaction models, and solves them by using domain-specific or general constraint propagation and branching methods (Apt, 2003). In this study, we focus on the modeling aspect and use a state-of-the-art commercial solver to demonstrate the effectiveness of our formulation. Using a general solver (either commercial or open source) is a practical and attractive option because it enables very rapid implementation of solution methods for a diverse set of optimization problems.

CP has been successfully used to solve a variety of problems in the domains of vehicle routing, scheduling, timetabling and others. See, for example, (Shaw, 1998), (Baptiste, 2003) and (Bukchin & Raviv, 2018). A CP model resembles an IP model in terms of syntax. It contains a declaration of decision variables with their domains, a set of constraints, and possibly an objective function. However, the CP modeling paradigm is much more expressive. In fact, the language is a superset of the integer linear programming modeling language. In addition to equality and inequality constraints between linear mathematical expressions, a CP model can contain nonlinear expressions, logical expressions, use decision variables as indices to other vectors of decision variables, and can include global constraints that capture a relationship between large sets of decision variables. There is no agreeable syntax for describing a CP model, and each solver implements a different modeling language that is suitable to express the type of constraint that it can propagate. Here, we are using the syntax inspired by the OPL language used by the ILOG CP Solver, a commercial package that we used for our numerical experiments. Some CP modeling constructs, used in our CP formulations below, are demonstrated in Appendix B.

Next, we present our CP model. As with the IP formulation, we will start with a model that minimizes the number of tardy jobs, which is equivalent to IP-U, and continue with a more general model that can optimize an arbitrary function over the completion times of the jobs. To this end, we introduce two sets of *interval* decision variables.

J_i Represents the scheduled time of job i - this is an optional interval with a predefined duration, minimal start time and maximal end time.

A_{ij} An auxiliary optional interval variable that is used to prevent jobs from overlapping on machines.

Model CP-U

$$\max \sum_{i=1}^{n} \text{presenceOf}(J_i) \tag{13}$$

Subject to:

$$\text{altenative}(J_i, \{A_{i1}, \dots, A_{i,m-s_i+1}\}) \qquad i = 1, \dots, n \tag{14}$$

$$\sum_{i=1}^{n} \sum_{k=\max(1,j-s_i+1)}^{j} \text{pulse}(A_{ik}, 1) \leq 1 \qquad j = 1, \dots, m \tag{15}$$

$$J_i \quad \text{interval, optional, length } p_i \text{ in } r_i, \dots d_i \qquad i = 1, \dots, n \tag{16}$$

$$A_{ij} \quad \text{interval, optional} \qquad i = 1, \dots, n, \; j = 1, \dots, m \tag{17}$$

The objective function (13) maximizes the number of scheduled jobs. Since the model allows jobs to be scheduled only if they satisfy their due date specifications, this objective function is equivalent to minimizing the number of tardy jobs. Constraint (14) states that if a job is scheduled (J_i is present), its respective auxiliary interval on one of the machines ($\{A_{i1}, \dots, A_{i,m-s_i+1}\}$) is also present and equal to J_i. Constraint (15) stipulates that the instantaneous load on each of the machines during the entire planning horizon never exceed one, which is to say that no two jobs overlap in terms of time and machines. The result of the summation on the left-hand side is a *pulse* that describes the utilization over time of each of the machines due to all the jobs. A job contributes to the utilization of a machine, j, at a given time if it is scheduled at this time and if its starting machine is one of the last s_i machines before machine j (and including it). In (16), the interval decision variable is defined as the optional interval of length p_i and range r_i to d_i, meaning that it can start no earlier than r_i and end no later than d_i. Finally, in (17), the A_{ij} variables are defined as optional intervals without further specifications.

Next, we present the generalized CP model, CP-G, which is equivalent to the IP-G model, (5)-(8). We use the same notation and very similar constraints, so in our discussion below, we will focus on the differences that allow this generalization.

Model CP-G

$$\min \sum_{i=1}^{n} f_i(J_i) \tag{18}$$

Subject to:

$$\text{altenative}(J_i, \{A_{i1}, \dots, A_{i,M-s_i+1}\}) \qquad i = 1..n \tag{14}$$

$\sum_{i=1}^{n} \sum_{k=\max(1,j-s_i+1)}^{j} \text{pulse}(A_{ik}, 1) \leq 1 \qquad j = 1..m, \qquad (15)$

$J_i,$ Interval, length p_i in $r_i, \ldots T \qquad i = 1..n \qquad (19)$

A_{ij}, Interval, optional $\qquad i = 1..n, j = 1..m \qquad (17)$

In the objective function, (18), we minimize the sum of a set of generic functions f_i over the properties of the interval of the jobs. In particular, the properties *presenceOf* and *endOf* are relevant. The function can be any expression that is available in the CP modeling language, including nonlinear ones, see an example below. In constraint (19), the definition of the interval is slightly different than in (16). Here, the jobs are not defined as optional (meaning they are mandatory) and their range is bounded from above by some upper bound on the length of the planning horizon, rather than by their due date. The upper bounds discussed for the IP-G model are valid here as well. This allows the generalization of scheduling a job after their due date (while bearing the implications in the objective function). Constraints (14), (15), and (17) have not changed.

For example, minimizing a weighted sum of the number of tardy jobs and the total tardiness, as in (9), can be obtained by substituting $f_i(J_i)$ with the following:

$$W^U(endOf(J_i) > d_i) + W^T(\max(0, endOf(J_i) - d_i))$$

Note that if our only goal is to minimize the number of tardy jobs, then model CP-U can be solved more efficiently, mainly since the domains that it specifies for the interval decision variables, J_i, are smaller.

4. Numerical experiments and analysis

In the previous section, the time-indexed integer programming (IP) and constraint programming (CP) formulations have been proposed for the consecutive multiprocessor job scheduling problem. Two variations were presented for each formulation, one for the objective of minimizing the number of tardy jobs (denoted as IP-U and CP-U) and the other for a more general setting (denoted as IP-G and CP-G). The general setting models were examined with a special case of lexicographically minimizing the number of tardy jobs and the total tardiness. This model, which is a special case of model G, is denoted in this section as model UT.

In addition to the IP and CP formulation, we examine, as a benchmark, a modified version of the mixed integer programming formulation originally presented in (Guan & Cheung, 2004), denoted as GC. Since, to the best of our knowledge, the model proposed here has not been studied in the literature, we adapt the (Guan & Cheung, 2004) formulation, which is based on continuous variables for the position of the jobs and integer variables for non-overlapping constraints. Then, we modify the objective of the original formulation, first for minimizing the number of tardy jobs (denoted CG-U) and then for the lexicographic problem (denote as CG-UT). Note that this formulation is also similar to other formulations suggested for the facility layout problems, see (Heragu, 2008). Our adaptation is presented in Appendix A.

Next, we compare the performance of the three formulations, for each of the problem types, presented above. Other than the general relative performance of the formulations, we study the effect of the problem parameters on the performance and identify the classes in which each formulation performs best.

4.1. Experimental design

A full factorial design experimentation was conducted. The instances for each of the two problems (denoted as U and G) were solved using the three formulations (GC, IP, and CP). The factors in our design are as follows:

1. The number of machines, m, with two levels of 15 and 25;
2. The number of jobs, n, with five levels of 10, 20, 30, 40, and 50;
3. The time granulation of the job's processing time. For each job i, a process time, p_i, is drawn from a discrete uniform distribution $U(\frac{p_{max}}{5}, p_{max})$, where p_{max} is a parameter responsible for the time granulation. Five levels were given to this parameter: 5, 10, 20, 40 and 80. For example, $p_{max} = 5$ indicates that the processing time is taken from a discrete distribution with possible discrete values of 1 to 5. When $p_{max} = 40$, for example, a higher granulation processing time is considered, which is taken from a discrete distribution with possible discrete values of 8 to 40.
4. The job size, namely, the number of consecutive machines it requires. The size of job i, s_i, is drawn from discrete uniform distribution $U(1, \lfloor s \cdot m \rfloor)$, where $\lfloor x \rfloor$ is the largest integer that is smaller than or equal to x, and the factor s has two levels of 0.3 and 0.5.

5. Two tightness levels of the window between the ready time and the due dates as described below.

The factors and their levels are depicted in Table 1. In summary, the full factorial experiment results in $2 \times 5 \times 5 \times 2 \times 2 = 200$ instances for each of the six models, 1200 runs in total. The solvers' running time was limited to 10 minutes but in most of the instances the solution process was terminated with a provably optimal solution earlier.

The ready times and due dates of the jobs were set to maintain two structures of the problems, with different levels of tightness. The values of the tightness parameters were determined via preliminary experiments, to avoid extreme cases of none or too many tardy jobs; cases that were found to be very easy to solve. Consequently, the structure with the high tightness resulted in an average of around 25% tardy jobs, while the structure with the low tightness resulted in an average of around 5% tardy jobs. The procedure of setting the corresponding parameters was done as follows. The ready time of each job i, r_i, was randomly drawn from the set $\{0,1, \dots, \Delta_{max}\}$ and the due date was taken from the set $\{r_i + p_i, r_i + p_i + 1, \dots, r_i + p_i + \Delta_{max}\}$, while keeping the value of Δ_{max} proportional to the capacity required by the scheduled jobs. Specifically, we set $\Delta_{max} = \zeta \bar{p} \cdot \bar{s} \cdot n/m$, where $\bar{p}$ is the average processing time and $\bar{s}$ is the average job size. The parameter ζ represents the level of tightness, with values of 0.3 for the tight instances, and 0.6 for the loose ones.

The length of the planning horizon in the UT problem was set based on the upper bound obtained from (11) and (12), using the best solution obtained for the equivalent U model.

Table 1: Factors summary

Factor		Set of possible values
m	Number of machines	$\{15, 25\}$
n	Number of jobs	$\{10, 20, 30, 40, 50\}$
p_{max}	Max. process time (resolution)	$\{5, 10, 20, 40, 80\}$
s	Job size factor	$\{0.3, 0.5\}$
ζ	Tightness level of the instance	$\{0.3, 0.6\}$

The experiments were conducted on an Intel i7-6700K desktop with 64 GB Ram running under Windows 10. We use Cplex Studio 12.8 to formulate and solve all the six models. The input files, the Python code that was used to generate it, and the OPL models are available upon request from the second author.

4.2. General results

First, we present the general results based on the entire set of the instances. The results are given in Table 2. When comparing the formulations, we distinguish between instances for which an optimal solution was found and those that consumed the entire time limit, resulting in approximated solutions. The former cases are evaluated by their runtime, while the latter are evaluated by their solution value. Other than comparisons based on specific (solution and runtime) measures, we developed a general lexicographic comparison rule. A formulation X is said to *lexicographically outperform* formulation Y with respect to a given instance if (1) it provides a lower solution value. (2) In case of a tie in (1), it provides a lower runtime. (3) In case of ties in (1) and (2), it provides a lower optimality gap. In the rare cases of ties in all the three criteria the two formulations are said to lexicographically outperform each other, i.e., this is a weak relation. As a result, for some instances there may be more than one best formulation.

For model U (minimizing the number of tardy jobs), we evaluate the performance of the three formulations, presented in columns CP-U, IP-U and GC-U. Based on the lexicographical preference, we can see that IP-U provides the best results for 131 instances, CP-U for 42 instances and GC-U for 31 instances. Ties (four in this case) are counted twice, leading to a total number that exceed the number of instances. The same ranking of the three formulations is obtained when conducting the pair comparison with a superiority of the IP (better than CP in 140 cases, versus 59 cases in the opposite direction, and better than GC in 148 cases versus 51 cases in the opposite directions).

When examining the solution values obtained within the time limit, we can see that CP-U and IP-U outperform GC-U, (with a slight difference between them), where they provide the best values in 181 instances and 178 instances, respectively, versus only 123 instances for the GC-U. The Pair comparison reveals the same pattern, as CP and IP outperform GC in 71 and 66 instances, respectively, while much smaller values, (2 and 16, respectively) are obtained in the opposite direction.

Table 2: Summary of the numerical experiment results

	Measure	CP-U	IP-U	GC-U	CP-UT	IP-UT	GC-UT
Lexicographic ranking	Count rank 1st	42	131	31	91	63	46
	$\succ_{lex}$ CP (Count)	==	140	56	==	79	74
	$\succ_{lex}$IP (Count)	59	==	51	121	==	106
	$\succ_{lex}$GC (Count)	133	148	==	126	94	==
Solution value	Best solution	181	178	123	100	87	65
	Number optimal	151	161	117	80	76	72
	Optimal proven	103	147	87	70	76	60
	Avg. optimality gap	27.70%	12.86%	49.94%	65%	31.63%	57.2%
	Avg. regret	0.1	0.4	0.8	35	7979	898
	Max. regret	2	7	8	1108	162482	13616
	$\succ_{sol}$ CP (Count)	==	17	2	==	34	20
	$\succ_{sol}$IP (Count)	22	==	16	90	==	71
	$\succ_{sol}$GC (Count)	71	66	==	102	61	==
Runtime	Avg. runtime (sec.)	315.0	194.1	350.2	407.5	579.2	408.7
	Median runtime (sec.)	311.5	147.7	600.0	600.0	600.0	600.0
	Avg. runtime (79/65 mutually optimal)	7.29	20.37	16.28	98.34	43.63	62.51
	Median runtime (79/65 mutually optimal)	1.35	0.24	0.78	11.42	4.11	1.62

When the number of optimal solutions are examined, IP again performs better than the other two, as it solves 161 instances to optimality (147 proven optimal), versus 151 (103) for CP and 117 (87) for GC. Not surprisingly, we can see that the average optimality gap of IP (12.86%) is much smaller than that of CP (27.70%) and GC (49.94%). For examining the robustness of the formulations' performance, we define the *regret* of formulation X for some instance as the difference between its solution and the best solution obtained by any of the formulations. This measure represents the loss occurs from choosing a certain formulation rather than the best formulation for some instance. Clearly, a zero-regret value indicates that the formulation obtains the best solution for that instance. One can see that CP has the lowest average regret of 0.1 tardy jobs, with a maximal value of 2. Namely, for *all* cases where CP trailed other formulations, the difference was at most two tardy jobs. IP and GC, on the other hand, have an average regret value of 0.4 and 0.8 tardy jobs, with a maximal value of 7 and 8, respectively.

To complete the analysis, we examine the runtime performance, where we distinguish between the runtime of all instances and the runtime of instances that were solved to optimality by *all* formulations (*mutually optimal*). For the U problem, 79 mutually optimal instances were identified. When all instances are considered, one can see that IP-U outperforms the other two formulations providing a much smaller average runtime of 194.1 seconds, versus 315.0 for CP-U and 350.2 for GC-U. Same ranking is obtained for the median, however, where the median is smaller than the average for IP and CP, it is not the case for the GC, which provides a median of 600 second. This indicates that more than half of the problems solved by the GC used all the allotted runtime.

When the lexicographic objective is considered (minimizing the number of tardy jobs and the total tardiness, as a primary and a secondary objective, respectively – see columns CP-UT, IP-UT, and GC-UT in the table), the CP performs much better, and outperforms IP and GC in most performance measures. Considering the lexicographic preference, the CP is counted first in 91 instances, versus 63 for the IP and 46 for the GC. In the pair comparison, the CP is much better than the other two formulations, while the GC is slightly better than the IP.

When the solution value is examined, the same ranking is obtained, both for the number of best solution obtained in each formulation (100 for the CP, 87 for the IP and 65 for the GC), and for the number of optimal solutions obtained for each formulation (80 for the CP, 76 for the IP and 72 for the GC). When only proven optimal solutions are considered, the IP performs better than the other two formulations, as all 76 optimal solutions are proven, versus 70 out of 80 for the CP and 60 out of 72 for the GC. This is probably since the IP provides better bounds than the other formulations, with an average optimality gap of 31.16% versus 57.2% for the GC and 65% for the CP. The last performance measure associated with the solution value is the average and max regret, explained above. As for the U formulations, the CP is significantly outperforms the other formulations also in UT, both for the average and maximal regret. This measure is highly important, because it indicates that even when the CP does not provide the best solution, the amount of loss versus the best formulation for that instance is relatively low.

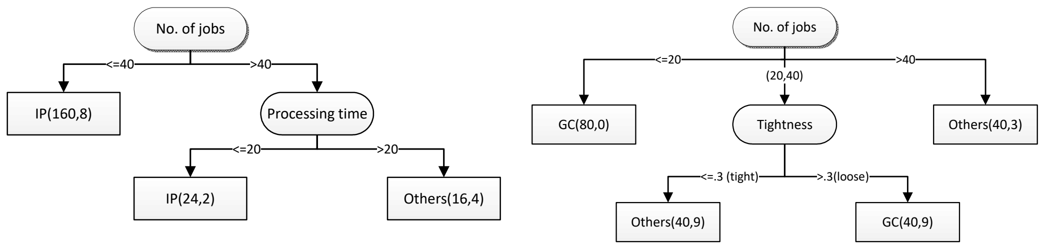
When considering the runtime, we distinguish again between the whole set of instances and the set of *mutually optimal* instances. One can see that in the first case, the CP and GC outperforms IP with respect to the average runtime, however, all the formulation provide a median value of 600 second. This is because the UT problem is harder to solve than the U problem, and in many of the instances require the maximal runtime in all formulations. Interestingly, when the mutually optimal instances are considered (65 out of 200), the IP provides the lowest average runtime, while the GC has the lowest average median. This is probably because the CP requires in general large runtime when providing an optimal solution, due to the large optimality gap, versus the other formulations.

To conclude the general results, we can see that the IP performs best when the U problems are considered. Still, when considering the robust performance, the CP is recommended, with a much smaller average and maximal regret. When considering the UT problems, the CP provides the best performance, both when the solution value, runtime and robustness measures are considered.

4.3. The effect of the problem parameters on the performance

In this section, we examine the effect of the problem parameters on the relative performance of the three formulations to design a recommended scheme for choosing the best formulation(s) to use for each combination of parameters. This problem is actually a classification problem, where the aim is to provide a prediction model to suggest the best formulation to use for each given instance. To this end, we have chosen to adopt a decision tree approach solve by CHAID (Chi-square automatic interaction detection) algorithm (Kass, 1980). The algorithm was executed on SPSS software.

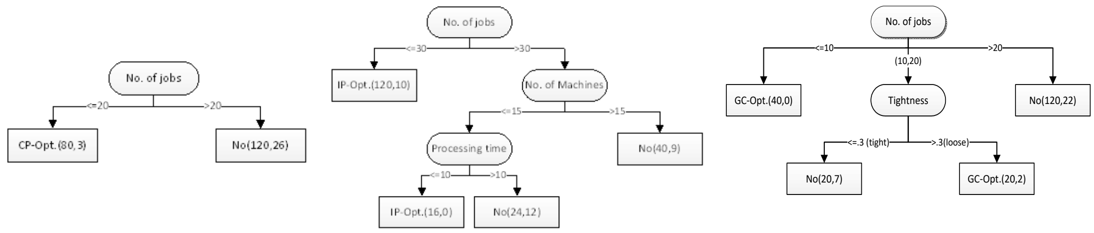
First, a decision tree was constructed to compare each formulation with the others. Hence, when examining a particular formulation, the purpose is to predict whether this formulation will provide the best solution (among the three formulations) for a given parameter combination, or not. The runtime limit was set to 10 minutes in this experiment. The results are shown in Figure 3, where the tree on the left refers to the IP model and the tree on the right to the GC model. Each inner node in the tree indicates a split based on the values of some parameter, and each leaf refers to the recommended decision for this set of parameters (class); namely, whether to use the examined formulation ('IP' or 'GC') or not ('Others'). In the parenthesis of each leaf, we can see the number of instances of this branch on the left and the number of classification mistakes on the right. The decision tree of the CP model was not presented here for the simple reason that it contains a single leaf; in 181 out of 200 instances, the CP provided the best solution (not necessarily alone), and no classification was identified for the rest. The decision tree of the IP shows that when the number of jobs is smaller than or equal to 40, IP is selected regardless of the values of the other factors (provides the best solution in 152 out of 160 instances). When the number of jobs is larger than 40, the selection of the IP depends on the granulation of the processing time, as it is selected when the granulation factor is smaller than or equal to 20 (best in 22 out of 24 instances). This result is not surprising due to the discrete time periods of the IP. When looking at the obtained decision tree of the GC, one can see that this approach provides the best solution only in relatively easy cases; namely, when the number of jobs is smaller than or equal to 20 (80 out of 80 instances), or when the problem is loose and the number of jobs is between 20 and 40 (31 out of 40). When the problems are tight or when the number of jobs exceeds 40, the other approaches outperform GC.



(a) IP versus others (b) GC versus others

Figure 3. Decision tree for comparison among formulations

In the second analysis, a decision tree was constructed to examine the effect of the problem parameters on the solution quality. For each instance, we indicated whether a validated optimal solution was obtained, or not. The decision trees depicted in Figure 4 show that IP outperforms the other two formulations, as it is expected to yield optimal solutions in cases including 136 instances (number of IP-Opt. in Figure 4b), versus only 80 in the CP (number of CP-Opt. in Figure 4a) and 60 in the GC (number of GC-Opt. in Figure 4c). More specifically, the main factor affecting ability to provide validated optimal solution is the number of jobs, as CP, IP and GC mostly provide optimal solutions when this number is smaller than or equal to 20, 30 and 10, respectively. Note that the CP in this case is inferior to the IP due to its relatively poor lower bounds. While only the number of jobs affects the CP, the other formulations are affected by other factors. The IP is expected to provide optimal solutions also for instances of more than 30 jobs, if the number of machines is smaller than or equal to 15, and the processing time granulation factor does not exceed 10. The GC is more limited in providing optimal solutions, as it is not recommended when the number of jobs exceeds 20, or even when this number is between 10 and 20 and the problem is tight. To conclude, when the number of jobs is relatively low, any of the formulations is expected to provide optimal solutions within the allotted time. When the number of jobs is between 20 and 30, only IP is expected to provide optimal solutions, regardless of the values of the other parameters. Even when the number of jobs exceeds 30, IP is expected to yield optimal solutions, still, subject to additional conditions.



(a) CP proven optimal (b) IP proven optimal (c) GC proven optimal

Figure 4. Decision tree for solution quality

5. Case study

In this section, we present a case study based on real data from a R&D lab of a large semiconductor company. This case study enables us to implement our approach in a dynamic environment, and compare its performance with the current practice. The lab uses *emulators* for testing new integrated circuit designs. Each emulator consists of five boards (i.e., machines), and five emulators are located in a sequence, constructing a structure of 25 consecutive boards. Each job requires one or more consecutive boards. The real-world problem is dynamic, where scheduling decisions are taken upon the arrival of new jobs to the system. Namely, each time a customer applies with a new job, this job is scheduled, and the customer is notified accordingly.

According to the current policy, each new arriving job is scheduled at the earliest possible start time. In case of multiple possible starting boards, the job is scheduled on the smallest indexed board. Once a schedule of a job is determined, this scheduled is final, and no rescheduling of this job is considered later on.

We propose integrating our IP and CP formulations in this dynamic environment, using a rolling horizon scheme. Namely, each time a customer applies to the system with a new job(s), the scheduling problem is solved by using one of our mathematical formulations, subject to the current state of the system, and commitments already made to previous customers. The formulation is solved each time with a lexicographic multi-objective function, with the number of tardy jobs as the primary objective and the total tardiness as the secondary one. In cases of ties, jobs are scheduled at the earliest possible time.

Once the model is solved, a commitment to the customer is made regarding the completion time. Two types of commitments are considered: (1) exact completion time of the newly scheduled job(s), as obtained by the model solution; and (2) satisfying the requested due date, if possible, or the completion time obtained from the model, otherwise. The former type is denoted by *exact commitment* (EC), and the latter by *due-date commitment* (DC). Note that the due-date commitment is looser than the exact commitment since some slack may occur in this case between the obtained completion time and the requested due-date. This slack may be used later on in rescheduling the existing jobs upon the arrival of new jobs. Rescheduling is also possible in the exact commitment; however, it is limited to the identity of the machines assigned to the job.

Our CP and IP models were modified to capture the nature of the dynamic environment. To this end, three types of jobs are defined upon the arrival of new jobs at the system, as follows: (1) jobs in progress (running jobs); (2) already scheduled jobs – not yet running; and (3) newly arriving job(s). The models are solved, starting from the current time with additional constraints that stipulate the schedule of the type 1 jobs and impose hard due-dates upon type 2 jobs, based on the commitment type (EC or DC). The type 3 jobs are freely scheduled. The two modified models are presented in Appendix C.

Our case study was based on historical data obtained from the R&D lab. We created a simulation environment that allows for the examination of the current scheduling policy and the rolling horizon policy using our formulations. The numerical study was performed using three log files of customers' requests. The log files contain information on the arrival time, the requested processing time and the number of boards required by each job. The ready times were taken as the arrival times of the jobs. The due date property of the jobs was missing, so it was randomly generated in between the minimal possible completion time ($r_i + p_i$) and the end the current week. According to the lab manager, these assumptions are in line with the current practice in the lab. The processing times and due dates in the log file are specified in units of half hours and there are cases where several jobs became available simultaneously. Each log file contains 171-288 jobs to be processed within a horizon of 37-60 days (1776-2880 time units). The characteristics of the datasets are given in Table 3.

Two runtime limits of 30 and 600 seconds were decided for each run. Each value may correspond to a different real-world situation, which is based on the urgency of the feedback to the customer. Note that applying the formulation in a rolling horizon scheme results in a myopic heuristic procedure, so their performance is not necessarily better than the existing greedy policy. Moreover, while increasing the runtime improves the accuracy of the obtained solution at each run, it does not necessarily improve the schedule.

Table 3. Characteristics of the datasets.

	No. of jobs	Planning horizon	Avg. (STD) process time (hours)	Min (Max) process time (hours)	Avg. (STD) job size (boards)	Min (Max) job size (boards)
Dataset 1	288	60 days	4.5 (9.9)	0.5 (168)	12 (4.6)	1 (25)
Dataset 2	196	37 days	6.7 (5.2)	0.5 (24)	12 (5.5)	1 (25)
Dataset 3	171	53 days	14.8 (5.2)	0.5 (72)	13 (5.3)	1 (25)

Table 4 presents the number of tardy jobs and the total tardiness of the three datasets for the current policy as well as for the two formulation based algorithm (CP and IP) under the two commitment types (EC, DC). In general, one can see that even for the lower time limit of 30 seconds, the procedure using the CP formulation outperformed the current policy in all cases. Although a significant improvement was obtained for the cases of the EC commitment type, a much larger improvement was achieved when the DC commitment type was considered, due to the further flexibility in the rescheduling process. For example, the number of tardy jobs was reduced from 22 to 8 in dataset 1, from 24 to 1 in dataset 2, and from 27 to 11 in dataset 3. When the

IP formulation was used, mixed results were obtained; although a significant improvement was achieved in three cases, inferior results were obtained for two cases (64 tardy jobs versus 27 in the current policy), and in one case, no feasible solution was reached. This leads to the conclusion that the CP is much more efficient for short runtimes. When the runtime of the formulation was increased to 600 seconds, the performance of the IP was improved, while the CP remained almost the same. As a result, both the CP and IP based procedures provided much better results (and comparable to each other) than the current policy.

To conclude, we have shown that using optimization techniques in a rolling horizon framework enables a significant improvement over the current policy in terms of both tardy jobs and total tardiness. The improvement is mostly related to the fact that the optimization models plan better in advanced and exploit the flexibility once new information is available. Although both IP and CP were found superior to the current policy, when fast feedback is required, we would recommend using the CP, due to its robust performance under a tight time limit.

We note that while in this simulation study it was assumed that the ready time of the jobs is known only upon arrival, allowing preorders of jobs, i.e., providing the ready time and due date information some time in advance, is desirable. The availability of this information may further increase the opportunity to plan ahead of time and improve the service provided by the system using the same resources. We recommend implementing preorders together with the implementation of the proposed scheduling mechanism.

Table 4. Case study results: number of tardy jobs (total tardiness)

	Current Policy	Commitment Type	CP (30 Sec.)	IP (30 Sec.)	CP (600 Sec.)	IP (600 Sec.)
Dataset 1	22(2731)	EC	17(2759)	17(2806)	17(2957)	17(2306)
		DC	8(621)	N/A	8(621)	8(651)
Dataset 2	24(1280)	EC	8(829)	9(954)	8(712)	9(954)
		DC	1(12)	1(1)	1(12)	1(1)
Dataset 3	27(3044)	EC	16(2536)	64(34016)	16(2536)	14(1823)
		DC	11(1650)	64(34016)	11(1650)	12(1799)

6. Conclusion

In this study, we introduced two effective optimization models for the consecutive multiprocessor job scheduling problem. We studied the objective function of minimizing the number of tardy jobs as well as minimizing a weighted sum of the number of tardy jobs and total tardiness. However, our model is amenable to a rich set of objective functions, including nonlinear and non-convex ones.

The effectiveness of our model was demonstrated by an extensive numerical study, and its applicability to real-life problems was demonstrated by embedding a rolling horizon version of the algorithm in a simulation of an actual on-line production environment. We showed that both the number of tardy jobs and the total tardiness could be reduced significantly.

While both our time-indexed IP and CP models were shown to be more effective on average than our adaptation of the GC formulations known from the literature, none of the three tested methods completely dominates the others. The IP model is good for obtaining optimal solutions quickly when the time granularity is rough, which we believe is the case in many actual manufacturing environments. It also provides a strong lower bound for the solutions, which is a nice theoretical trait that is not necessarily important for practitioners. The CP models delivered the best solutions in most of our experiments and was found the most robust approach, while lags behind the other methods by a tiny gap in the few cases when it did not find the best solution. The GC models are insensitive to the time granularity and thus can be useful when the accuracy of the time input is high. However, the lower bounds established by the GC model are very weak.

There are several interesting directions for future research based on the results of this study. In particular, the effectiveness of time-indexed integer formulations and constraint programming formulations as practical solution methods for other variants of the multiprocessor job scheduling problem should be verified. Since time-indexed formulations are especially effective when the time granulation is low, this naturally leads to two step heuristic methods in which the sequence of the jobs is decided based on a model with a rough time granulation, after rounding the original input, and the schedule is then fine-tuned to respect the accurate times. The effect of rough time granulation is of particular interest in an online situation when tentative decisions about the future need to be made quickly.

References

Agra, A., & Oliveira, M. (2018). MIP approaches for the integrated berth allocation and quay crane assignment and scheduling problem. *European Journal of Operational Research, 264*, 138-148.

Apt, K. Y. (2003). *Principles of Constraint Programming* (1 ed.). Amsterdam: Cambridge University Press.

Baptiste, P. (2003). A note on scheduling multiprocessor tasks with identical processing times. *Computers & Operations Research, 30*, 2071-2078.

Bianco, L., Blazewicz, J., Dell'Olmo, P., & Drozdowski, M. (1995). Scheduling multiprocessor tasks on a dynamic configuration of dedicated processors. *Annals of Operations Research, 58*, 493-517.

Bierwirth, C., & Meisel, F. (2010). A survey of berth allocation and quay crane scheduling problems in container terminals. *European Journal of Operational Research, 202*, 615-627.

Bukchin, Y., & Raviv, T. (2018). Constraint programming for solving various assembly line balancing problems. *Omega, 78*, 57-68.

Chen, J., & Lee, C. Y. (1999). General multiprocessor task scheduling. *Naval Research Logistics, 46*, 57-74.

Ernst, A. T., Oguz, C., Singh, G., & Taherkhani, G. (2017). Mathematical models for the berth allocation problem in dry bulk terminals. *Journal of Scheduling, 20*, 459-473.

Guan, Y., & Cheung, K. R. (2004). The berth allocation problem models and solution methods. *OR Spectrum*, 75-92.

Heragu, S. (2008). *Facilities Design.* CRC Press.

Huang, J., Chen, J. C., & Wang, J. (2007). A simple linear time approximation algorithm for multiprocessor job scheduling on four processors. *Journal of Combinatorial Optimization, 13*, 33-45.

IBM. (2009). *Detailed Scheduling in IBM ILOG OPL with IBM ILOG CP Optimizer - Tutorial.* IBM Corp.

Imai, A., Chia, C. H., Etsuko, N., & Papadimitriou, S. (2008). The simultaneous berth and quay crane allocation problem. *Transportation Research Part E*, 900-920.

Imai, A., Nishimura, E., & Papadimitriou, S. (2001). The dynamic berth allocation problem for a container port. *Transportation Research Part B: Methodological, 35*, 401-417.

Imai, A., Sun, X., Nishimura, E., & Papadimitriou, S. (2005). Berth allocation in a container port using a continuous location space approach. *Transportation Research Part B, 39*, 199-221.

Kass, G. (1980). An exploratory technique for investigating large quantities of categorical data. *Journal of the Royal Statistical Society: Series C (Applied Statistics), 29*(2), 119-127.

Leonardi, S., & Raz, D. (2007). Approximating total flow time on parallel machines. *Journal of Computer and System Sciences, 73*, 875–891.

Martello, S., Monaci, M., & Vigo, D. (2003). An exact approach to the strip-packing problem. *INFORMS Journal on Computing, 15*(3), 310-319.

Moon, K. (2000). A Mathematical Model and a Heuristic Algorithm for Berth Planning. *Ph.D. Thesis, Pusan National University, Pusan.*

Park, Y. M., & Kim, K. H. (2005). A scheduling method for berth and quay cranes. *Container Terminals and Automated Transport Systems*, 159-181.

Pinedo, M. L. (2012). *Scheduling: Theory, Algorithms, and Systems, Fourth Edition.* New York: Springer.

Sevaux, M., & Thomin, P. (2001). Heuristics and metaheuristics for a parallel machine schedulign problem: a computational evaluation. *Proceedings of the 4th Metaheuristics International Conference*, (pp. 411-415).

Shaw, P. (1998). Using constraint programming and local search methods to solve vehicle routing problems. *International Conference on Principles and Practice of Constraint Programming.* (pp. 417-431). Berlin Heidelberg: Springer.

Umang, N., Bierlaire, M., & Vacca, I. (2013). Exact and heuristic method to solve berth allocation problem in bulk ports. *Transportation Research Part E, 54*, 14-31.

Appendix A - Relative position MILP formulation

In this appendix, we present a relative position MILP formulation to our problem adopted from (Guan & Cheung, 2004) with minor modifications to reflect our objective function. The original formulation minimizes the weighted flow time. Similar relative position formulations were presented in (Moon, 2000) and (Imai, Sun, Nishimura, & Papadimitriou, 2005) for the berth allocation problem and in (Heragu, 2008) for the facilities layout problem. It is presented here since we used it to benchmark our integer programming and CP formulation.

We use the same notation for the model parameters as presented in Section 3 and introduce the following decision variables:

x_i start time of job i

y_i first machine occupied by job i

$\sigma_{ij} = \begin{cases} 1, & \text{if job } i \text{ ends before job } j \text{ starts} \\ 0, & \text{otherwise} \end{cases}$

$\delta_{ij} = \begin{cases} 1, & \text{if job } i \text{ ends on a machine with smaller index than the first machine of job } j \\ 0, & \text{otherwise} \end{cases}$

$U_i = \begin{cases} 1, & \text{if job } i \text{ is tardy} \\ 0, & \text{otherwise} \end{cases}$

T_i The tardiness of job i

The model presented below implements a special case of our general IP and CP models, where the objective is to minimize a weighted sum of the number of tardy jobs and the total tardiness. The model presented below captures the case of minimizing the number of tardy jobs. Note, however, that other interesting objective functions, such as a nonlinear penalty for tardiness, are not directly represented by this model.

$$\min \left\{ W^U \sum_{i=1}^{n} U_i + W^T \sum_{i=1}^{n} T_i \right\} \tag{20}$$

Subject to:

$$U_i \times T \geq c_i - d_i \qquad \forall i = 1, \dots, n \tag{21}$$

$$x_j \geq c_i - (1 - \sigma_{ij})T \qquad \forall i, j = 1, \dots, n \tag{22}$$

$$y_j \geq y_i + s_i - (1 - \delta_{ij})m \qquad \forall i, j = 1, \dots, n \tag{23}$$

$$\sigma_{ij} + \sigma_{ji} + \delta_{ij} + \delta_{ji} \geq 1 \qquad \forall i, j = 1, \dots, n \tag{24}$$

$$\sigma_{ij} + \sigma_{ji} \leq 1 \qquad \forall i, j = 1, \dots, n \tag{25}$$

$$\delta_{ij} + \delta_{ji} \leq 1 \qquad \forall i, j = 1, \dots, n \tag{26}$$

$$c_i = p_i + x_i \qquad \forall i = 1, \dots, n \tag{27}$$

$$x_i \geq r_i \qquad \forall i = 1, \dots, n \tag{28}$$

$$c_i \leq T \qquad \forall i = 1, \dots, n \tag{29}$$

$$y_i \leq m - s_i \qquad \forall i = 1, \dots, n \tag{30}$$

$$T_i \geq c_i - d_i \qquad \forall i = 1, \dots, n \tag{31}$$

$T_i \geq 0$	$\forall i = 1, \dots, n$	(32)
$x_i \geq 0$	$\forall i = 1, \dots, n$	(33)
$y_i \geq 0, integer$	$\forall i = 1, \dots, n$	(34)
$\sigma_{ij} \in \{0,1\}, \delta_{ij} \in \{0,1\}$	$\forall i, j = 1, \dots, n$	(35)
$U_i \in \{0,1\}$	$\forall i = 1, \dots, n$	(36)

The objective function (20) minimizes the number of tardy jobs and the total tardiness. Constraint (21) enforces a value of one in the binary variable U_i if job i is tardy. Constraint (22) allows a value of one in σ_{ij} only if job i is completed before the beginning of job j. Similarly, Constraint (23) allows one in δ_{ij} only if the index of the first machine of job i is greater than the index of the last machine of job j. Constraint (24) eliminates overlapping of the processing of jobs on the machines. Constraints (25) and (26) state that jobs can either precede or succeed each other. These are valid inequalities that are not required for the definition of the feasible set of solutions but may tighten the formulation. Equality (27) is used to assign a value to the completion time. We could save both this constraint and variable by replacing c_i with $p_i + x_i$ but we kept it here to remain compatible with (Guan & Cheung, 2004). Equation (28) requires that the starting time of a job is not earlier than it ready time. Constraint (29) states that a job cannot be completed before the end of the predefined planning horizon (again presented here only for computability with the original formulation). Constraint (30) states that the index of the first machine of a job is such that the job can fit within the number of machines in the system. Constraints (31) and (32) enforce the correct value of tardiness for each job. The domains of the rest of the decision variables are specified in (33)-(36). In particular, the integrality of the first machine of a job is stipulated in (34), but we note that, even if this integrality requirement is omitted, the resulted solutions are integral. This is probably because, given the integral values of the parameters and the rest of the decision variables, the basic solutions of the relaxed model are all integers.

Appendix B – CP modeling constructs examples

Here, we describe several CP modeling constructs that we use in our CP models.

Reification – any valid logical expression including an equality or inequality may be used as an argument of an indicator function that returns a value of one if the expression is true and zero otherwise. For example, consider a vector of integer decision variables x_i, $i = 1..n$. The requirement that at least two $x's$ equal 23, can be expressed by the following constraint.

$$\sum_{i=1}^{n}(x_i = 23) \geq 2$$

Interval variable – an object that describes a continuous period in which a job of a given length is processed. It is useful for representing a solution to scheduling problems. An interval is characterized by its range and the length of the job to be scheduled in it. The solver assigns the start time of the job. An interval can be mandatory, meaning that the solver must assign a job to it, or optional. Several functions are defined to query information and to impose constraints on intervals. For example, given an interval variable a, the functions startOf(a) and endOf(a) return the start and end times of the job scheduled in interval a, respectively. The function presenceOf(a) returns one if a job is scheduled in the solution and zero otherwise. When defining an interval variable, it is possible to specify its domain.

Pulse - The function $pulse(a, h)$ obtains an interval a and an integer h as arguments. It returns a step function defined over the planning horizon (time). The value of the returned function is h between the start and the end of the interval and zero otherwise. The language also defines the summation of pulse functions and it is possible to set a bound on the maximal level of the pulse. For example,

$$\text{pulse}(a, 1) + \text{pulse}(b, 1) \leq 1$$

is a constraint that eliminates the possibility that the two intervals, a and b, will overlap.

Alternative - To instruct the solver to select one of several mutually exclusive optional intervals, $\{a_1, \dots, a_n\}$, as present and to set its start and end time to be identical to a mandatory interval, a, we can use the following statement.

$$\text{alternative}\left(a, \{a_{1,} \dots, a_n\}\right).$$

For additional information about the OPL language and its usage for the formulation of CP models, see (IBM, 2009). Note that while OPL and CPLEX CP solver are the propriety of IBM ILOG, other constraint programming packages support similar modeling tools.

Appendix C – The mathematical models for the rolling horizon framework

In this appendix, we present the models used in our case study to solve the problem in dynamic settings. These are adaptions of the IP-G and CP-G models. The models are repeatedly solved whenever new job(s) arrive at the system. In this version of the model, the beginning of the planning horizon is the current time, newly arrived jobs can be scheduled freely, and jobs that were previously scheduled but not started yet can be rescheduled subject to their due-date commitment. Jobs already in process are not considered directly, but their machine occupancy is represented by a ready-time parameter for each of the machines.

The cost parameter c_{it} was set such that the following objectives will be minimized lexicographically: (1) The number of tardy jobs; (2) Total tardiness; (3) Jobs start time; (4) Index of the first machine processing the job.

The purpose of the last two components (with the smallest weight) is to schedule the jobs close to each other in time and space (machines) to make room for future jobs to arrive.

The parameter d_i has a slightly different meaning in the models presented here. For previously scheduled jobs, it represents the time already committed to the customer. Its value depends on the commitment type, i.e., typically smaller for the EC commitment type and larger for the DC commitment type.

Next, we introduce some new notation to be used by the models:

N	Set of all the jobs that have not started yet, including the new ones.
ρ_j	Ready time of machine j, based on the schedules of jobs that have already started.

δ_i The latest possible start time of job i, which is equal to $T - p_i + 1$, for jobs and $d_i - p_i + 1$, for jobs with previously committed due dates.

$$\min \sum_{i \in N} \sum_{j=1}^{m-s_i+1} \sum_{t=\max(r_i,\rho_j)+1}^{\delta_i} c_{it}\ x_{ijt} \tag{37}$$

Subject to:

$$\sum_{j=1}^{m-s_i+1} \sum_{t=\max(r_i,\rho_j)+1}^{\delta_i} x_{ijt} = 1 \qquad \forall i \in N \tag{38}$$

$$\sum_{i \in N} \sum_{\substack{j'= \\ \max\{1, j-s_i+1\}}}^{j} \sum_{\substack{t'= \\ \max\{r_i+1, t-p_i+1\}}}^{\min\{\delta_i, t\}} x_{ij't'} \leq 1 \qquad j = 1, \ldots, m, t = 1, \ldots, T \tag{39}$$

$$x_{ijt} \in \{0,1\} \qquad \forall i \in N,\ j = 1, \ldots, m - s_i + 1,\ t = 1, \ldots, \delta_i \tag{40}$$

The objective function (37) minimizes the total cost of the schedule. Constraint (2) stipulates that all the jobs be scheduled to start at a valid time. Constraint (7) assures that jobs do not overlap each other in time or space. Constraint (40) defines the domain of the decision variable.

The CP-G model remains unchanged except for the domain of the interval decision variables that are now defined as follows:

J_i , Interval, length p_i in $r_i, \ldots \delta_i$ $\qquad i = 1, \ldots, n$ (41)

A_{ij}, Interval, optional, in $\rho_j, \ldots, \delta_i$ $\qquad i = 1, \ldots, n, j = 1, \ldots, m$ (42)